Hence

$$\chi^2 = \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A - t_i - p_s)^2 \tag{2}$$

The reliability coefficient of a test is the ratio of the variance of the "true scores" to the variance of the obtained scores, or in other words, gives the percentage of the obtained variance that may be spoken of as "true" variance or not due to the unreliability of the test. If we let σ^2_t represent the variance of the obtained scores and σ_d^2 the discrepancy between the variance of the obtained scores and the variance of the "true" scores, the reliability coefficient is given by the ratio: $r_{tt} = (\sigma^2_t - \sigma^2_d)/\sigma^2_t$. The variance of the error term, y_{is}, or σ^2_d, is the expression of which we wish to obtain the best estimate. According to Markoff's theorem,* the best linear estimate of y_{is} can be obtained, in the situation where the σ's of the independent observations are all equal, by minimizing the sum of squares (2) with respect to A, t_i, and p_s as independent variables and substituting these values, A°, t_i°, and p_s°, in (1) to give y_{is}', the best linear estimate of the discrepancy between the obtained score and the "true" score.

A first necessary condition for minimizing (2) is that the partial derivatives with respect to A, t_i and p_s must vanish.

$$\frac{\partial \chi^2}{\partial A} = -2 \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A - t_i - p_s) ; \tag{3}$$

$$\frac{\partial \chi^2}{\partial t_i} = -2 \sum_{s=1}^{k} (X_{is} - A - t_i - p_s) \quad \text{for each } i = 1, 2, \cdots, n ; \tag{4}$$

$$\frac{\partial \chi^2}{\partial p_s} = -2 \sum_{i=1}^{n} (X_{is} - A - t_i - p_s) \quad \text{for each } s = 1, 2, \cdots, k . \tag{5}$$

Setting each of these partial derivatives equal to zero and solving simultaneously gives the values for A, t_i, and p_s which minimize χ^2. These values of A, t_i, and p_s which render χ^2 a minimum will be designated by A^0, t_i^0, and p_s^0.

$$A^0 = \frac{1}{nk} \sum_{i=1}^{n} \sum_{s=1}^{k} X_{is} - \frac{1}{n} \sum_{i=1}^{n} t_i^0 - \frac{1}{k} \sum_{s=1}^{k} p_s^0 ; \tag{6}$$

$$t_i^0 = \frac{1}{k} \sum_{s=1}^{k} (X_{is} - p_s^0) - A^0 \quad (i = 1, 2, \cdots, n) ; \tag{7}$$

* Neyman, J. The Markoff Method and Markoff Theorem on Least Squares. *Journal of the Royal Statistical Society*, 1934, **97**, 593-594.

$$p_s^0 = \frac{1}{n} \sum_{i=1}^{n} (X_{is} - t_i^0) - A^0 \quad (s = 1, 2, \cdots, k) \tag{8}$$

Substituting these values of A^0, t_i^0, and p_s^0 in (2) gives the minimum value of χ^2 which is designated by S_0.

$$S_0 = \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A^0 - t_i^0 - p_s^0)^2$$

$$= \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - \bar{x}_{i.} - \bar{x}_{.s} + \bar{x})^2 , \tag{9}$$

where

$$\bar{x}_{i.} = \frac{1}{k} \sum_{s=1}^{k} X_{is} \; ; \bar{x}_{.s} = \frac{1}{n} \sum_{i=1}^{n} X_{is} \; ; \quad \bar{x} = \frac{1}{nk} \sum_{i=1}^{n} \sum_{s=1}^{k} X_{is} . \tag{10}$$

Substituting these values of A^0, t_i^0, and p_s^0 in (1) gives the best linear estimate of y_{is} (i.e., the error component of the response of the s-th student to the i-th item).

Thus

$$y'_{is} = X_{is} - \bar{x}_{i.} - \bar{x}_{.s} + \bar{x} , \tag{11}$$

Since $\sum_{i=1}^{n} \sum_{s=1}^{k} y'_{is} = 0$, $S_0 = \sum_{i=1}^{n} \sum_{s=1}^{k} (y'_{is} - \bar{y})^2$, so that S_0 is f times the variance of the y'_{is}, the error component, where f is the number of degrees of freedom or the number of independent variates necessary to express the sum, S_0. It is clear that $f = (n-1)(k-1)$ if we consider the nk X_{is}'s arranged in n rows and k columns, so there are $(n-1)$ independent variates in each of the $(k-1)$ columns. Hence the best estimate of the variance of the error component is

$$\frac{S_0}{(n-1)(k-1)} . \tag{12}$$

In order to determine the best estimate of σ^2_s, the variance of the component p_s associated with the student, assume p_s to be normally distributed with variance σ^2_s. Since p_s is independent of y_{is}, $\sum_{i=1}^{n} \sum_{s=1}^{k} (y_{is} + p_s)^2$ is distributed as χ^2.

Then

$$\chi^2 = \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A - t_i)^2 ; \tag{13}$$

$$\frac{\partial \chi^2}{\partial A} = -2 \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A - t_i) ; \tag{14}$$